The Puzzle of the Replacement Ratio in the Context of Renewal Theory

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Abstract

The models Feldstein and Rothschild (1974) and Jorgenson (1974) adopted to highlight the nature of the replacement ratio were identical. Yet, even though the theorems they derived from them were complementary and reinforced each other, the authors reached diametrically opposite conclusions. Digging deeper into the controversy that erupted, it emerges that the staying power of the theorem, according to which replacement is a constant proportion of the outstanding capital stock, may be attributed to the following reasons. The discernible shift from realism to instrumentalism in the methodology of economics; Its operational advantages; The data that accumulated, thus facilitating research without having to compute capital stock series from scratch; The inertia of the status quo, which is sustained by the absence of a process to decide when a theorem is in conflict with experience and should be set aside, and lastly the lack of a model leading to a more useful theorem than the one under consideration. In this light it is concluded that the time has come for research efforts to be directed towards constructing and testing models in which the useful life of capital is determined endogenously in the presence of embodied technological change.

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1. Introduction

Much of what we know about the structure and stability of contemporary economies may be meaningfully related to certain key ratios. When Klein (1962, p.183) was writing, his list of great ratios included: the consumption-income ratio (propensity to consume), the capital-output ratio (acceleration principle), the labor's share of output (income distribution), the ratio of cash to income (reciprocal of velocity of circulation), and the capital-labor ratio (fixed factor proportions). From this account it follows that at that time economic theorists and econometricians conceived of investment as additions to the capital stock that were induced by changes in output through a Koyck type adjustment mechanism. Actually, as it may be ascertained from Haavelmo's (1960) treatise on the subject, there was no theory of gross investment, whereas the body of theory on replacement investment emanating from the seminal contributions by Hotelling (1925), Preinreich (1940), Terborgh (1949), Smith (1957) and others, was considered unsuitable to serve as microeconomic foundations for constructing a comparable aggregate theory.[1]

This disparate state in the theory of investment started to change with Smith (1961, p.166). In particular, to formulate a model of replacement investment based on rational choice, he postulated that the capital-using firm behaves as if to minimize:

$$C = (m + bT)x + (\hat{\delta} + aT + q / T + rq)K, \tag{1}$$

where the various symbols have the following meanings: $C =$ total current cost; $x =$ variable input like the amount of energy consumed; $K =$ stock of durable goods; $T =$ useful life of the stock of durable goods; $m =$ unit cost of variable input; $q =$ purchase cost of the stock of durable goods; $b =$ age related rate of deterioration in the usage of the variable input; $a =$ age related rate of deterioration in the services from the incumbent durables due to embodied technological change in newer vintages; $r =$ a constant rate of interest, and $\hat{\delta} =$ a constant non-age related proportional rate of deterioration in capital services. Looking closer at this expression observe that the efficiency of capital declines for three reasons. The first of them is that as capital ages it may require more inputs of materials, energy, maintenance, etc., in order to yield the original level of output. This effect constitutes the so-called *input decay* and is captured in the model by the age-related term bT.

The second reason has to do with *output decay* and springs from the observation that as capital ages it may become less efficient due to normal wear and tear. Even though this effect is age-related as well, in the model it is stipulated to be a proportion $\hat{\delta}$ of the outstanding capital stock. Finally, the third reason relates to technological change and implies that as the capital in place ages it becomes inferior relative to new capital that embodies the most recent advances in science and technology. This effect is identified as *technological obsolescence* and in the model it is approximated by the term aT. From this formulation it turns out that the only part of replacement investment that was conceived as proportional to capital stock was to counterbalance *output decay* and it was adopted only as a convenient mathematical approximation. Otherwise the model was very general because it accounted for losses in the efficiency of capital services from all possible sources of physical and economic deterioration.

Soon after this remarkable conceptualization of the fundamental replacement problem there appeared a highly influential paper by Jorgenson (1963) where in terms equivalent to (1) he demonstrated that:

$$C = mx + (\hat{\delta} + rq)K = mx + q(r + \delta)K \ . \tag{2}$$

But this restatement of the problem constituted a major break from all past endeavors in at least one crucial respect.[2] This was that, by abstracting completely from the impact of input decay and technological obsolescence and attributing all deterioration to output decay, which evolved at the constant proportional rate δ, replacement was rendered invariant with respect to the useful life, T. Understandably therefore the justifications that warranted this far-reaching departure from the received economic theory of replacement were of particular importance. In this regard, here is how Jorgenson supported his assertion that the rate of deterioration of capital services and hence of replacement investment is a constant proportion of the capital stock:

> "…The justification for this assumption is that the appropriate model for replacement is not the distribution of replacements of a single investment over time but rather the infinite stream of replacements generated by a single investment; in the language of probability theory, replacement is a recurrent event. It is a fundamental result of renewal theory that replacements for such an infinite stream approach a constant proportion of capital stock for (almost) any distribution

> of replacements for a single investment and for any initial age distribution of capital stock. This is true for both constant and growing capital stocks…" (p. 251).

Thus, based on the claim that it could be derived from renewal theory and the determination with which Jorgenson (1965) returned to support it empirically, the notion that replacement investment is a constant proportion of the outstanding capital stock begun to be accepted as a proposition of general validity.

However, at the same time, there started to appear evidence, which raised serious doubts as to whether this proposition applied in reality. In the United States, for example, such evidence was offered by Walker (1968) and Wykoff (1970), who looked into the scrappage and the price-age profiles of automobiles, respectively, and Feldstein and Foot (1971) and Eisner (1972), who investigated the variability of replacement investment-capital stock ratio in the sector of manufacturing. What these research efforts showed was that the replacement ratio varied systematically with changes in conventional economic forces. So the literature entered into a state of uncertainty because either Jorgenson's claims were unfounded or the evidence from the above empirical studies was marred by erroneous shortcomings.

In view of this ambiguity, theoretically oriented research efforts were expected to intensify.[3] True to this expectation, Feldstein and Rothschild (1972) turned their attention in this direction. As a result, until their discussion paper was published two years later, the tide seemed to be turning in favor of the view that a constant replacement ratio could be obtained from renewal theory under so restrictive conditions that it might hold in reality only by numerical accident. But in the same year Jorgenson (1974) came out roaring with a powerful defense of his earlier claims. In particular, he established that a constant replacement ratio could be derived from renewal theory under quite general conditions (henceforth to be referred to as the "theorem of proportionality" or just the "theorem") and ever since this result has influenced economic theory and policy as if the arguments put forth by Feldstein and Rothschild (1974) were irrelevant or misplaced. Thus what I wish to do here is to revisit that very important debate and try to set the record straight in light also of the findings in Bitros (2009a; 2009b), where I survey and assess the voluminous theoretical and empirical literature in this area.

To this effect the present paper is organized as follows. The first task is to preclude the possibility that the puzzle emanated from technical reasons. In doing so Section 2 looks at the models that were adopted in the two studies and ascertains that they are identical. Section 3 describes how the authors employed their models to obtain necessary and sufficient conditions for replacement investment to be proportional to the outstanding capital stock and assesses the standing of the theorems thus derived. Having excluded that the puzzle is due to flaws in the analyses, Section 4 turns for clues in other directions. In particular, it looks into the shifts that took place in the methodology of economics, the difficulties in formulating an alternative model of depreciation and replacement, i.e. one centered on the useful lives of durables goods, etc. Finally, Section 5 closes with a summary of the main findings and the conclusions.

2. The models in the two studies

Jorgenson (1974, pp. 211-213) assessed the empirical evidence that Feldstein and Foot (1971) had discovered against the theorem of proportionality and rejected it on the grounds that they had failed to define and measure the stock of capital consistently. But he ignored completely the results that Feldstein and Rothschild (1972/1974) had obtained using a model grounded in renewal theory.[4] From this observation one would be tempted to surmise that he did not find any fault with their model. On the other hand, after the appearance of Jorgenson's (1974) contribution, the latter authors did not care to revisit the puzzle that emerged, and hence one would be tempted again to surmise that they did not find any fault with his model either. Therefore, any attempt to reconcile their contradictory claims regarding the nature of the replacement ratio in the context of renewal theory must start with a description of the models in the two studies.

2.1 Rules, conventions and definitions of variables

On the way to this task, it is convenient to start with Table 1 below, which explains the rules, the conventions and the symbols used to denote the variables and the parameters in the two models. The rules and the conventions, which might affect the results, are shown in the top half of the table. From them it turns out that the only difference is in the length of time required for installed investment to become productive. In particular, notice that whereas in the Feldstein and Rothschild model (henceforth to be referred to as the F&R model) installed investment becomes productive in the next period, in Jorgenson's model

Table 1 Rules, conventions and symbols adopted in the two models

	Feldstein & Rothschild (1974)[1]	Jorgenson's (1974)
Rules and conventions		
Measurement of capital	Efficiency units	Efficiency units
Amount of services by a unit of capital in the first year of its life	1	1
Capital stock as a column vector[2]	$M(t)$	…
Investment becomes productive in period:	$t+1$	t
Symbols		
Stock of capital	$K(t)$	K_t
Percentage of surviving efficiency of vintage investment relative to the original	s_υ	d_τ
Surviving capital from vintage investment	$M_\upsilon(t)$	$d_\tau A_{t-\tau}$
Gross vintage investment[3]	$M_\upsilon(t)(1/s_\upsilon)$	$A_{t-\tau}$
Replacement investment	$R(t)$	R_t
Replacement investment-capital stock ratio	$r(t)$	δ_τ
Mortality distribution defined as:	…	$m_\tau = (d_{\tau-1} - d_\tau)$
Age structure of the capital stock	$\alpha(t)$	…

Notes: 1. Henceforth reference will be made to the published paper.

2. Its components are: $M_1(t)\ M_2(t)\ ...\ M_\upsilon(t)\ ...$

3. Note that, whereas $M_\upsilon(t)(1/s_\upsilon)$ denotes the amount of investment undertaken to replace the deterioration of the $M_\upsilon(t)$ vintage of investment at time *t*, $A_{t-\tau}$ denotes gross vintage investment.

(henceforth to be referred to as the J model) investment becomes productive as soon as it is installed. This difference though has to do only with the indexing of the installed vintages of investment and hence it leaves the results unaffected.

2.2 Assumptions

Table 2 describes the assumptions on which the two models are based. Looking downwards at the two extreme right columns, observe that in the F&R model durable goods last for V periods. On the contrary, in the J model they last forever since their useful life is set equal to ∞ and their scrappage is forced through the condition that d_τ tends to zero as τ tends to infinity. Could this difference be responsible for the puzzle

Table 2 Assumptions embedded in the two models

	Feldstein & Rothschild (1974)	Jorgenson's (1974)
Types of durables goods in the stock of capital	Homogeneous	Homogeneous
Source of deterioration of capital efficiency	Output decay	Output decay
Technological obsolescence	Ignored	Ignored
Decay function of vintage investment	Time invariant (s_υ)	Time invariant (d_τ)
Re-investment opportunities	Ignored	Ignored
Services surviving from vintage investment	$s_{\upsilon-1}M_{\upsilon-1}(t)$ $s_1 = 1,\ s_V = 0$ $\upsilon = 2,\ldots,V$	$d_\tau A_{t-\tau}$ $d_0 = 1,\ \lim_{\tau\to\infty} d_\tau = 0$ $\tau = 0,\ldots,\infty$

regarding the nature of the replacement ratio? The answer is no because drawing on Bitros and Flytzanis (2005) the puzzle has to do not with the possible differences in the level of the replacement ratios, but whether the replacement ratios that result from the two models are constant or variable.

2.3 Mathematical structure of the models

The main equations of the two models are shown in Table 3. Observe that with the exception of the difference mentioned above regarding the durability of capital goods, the definitional and behavioral equations are identical. However, while in the F&R model the

Table 3 Basic definitional and behavioral equations of the two models

	Feldstein & Rothschild (1974)	Jorgenson's (1974)
Stock of capital services	$K(t) = \sum_{\upsilon=1}^{V} M_\upsilon(t)$	$K_t = \sum_{\tau=0}^{\infty} d_\tau A_{t-\tau}$
Replacement investment	$R(t) = \sum_{\upsilon=1}^{V} M_\upsilon(t)(1-s_\upsilon)$	$R_t = \sum_{\tau=1}^{\infty} (d_{\tau-1} - d_\tau)A_{t-\tau}$
Age structure of the capital stock	$a(t) = \frac{1}{K(t)} M(t)$	…
Replacement ratio	$r(t) = \frac{R(t)}{K(t)} = \sum_{\upsilon=1}^{V} \alpha_\upsilon(t)(1-s_\upsilon)$	$\hat{\delta} = \frac{R_t}{K_{t-1}} = \sum_{\tau=1}^{\infty} \delta_\tau \frac{K_t - K_{t-\tau-1}}{K_{t-1}}$

replacement ratio $r(t)$ is expressed in terms of the age structure of the capital stock and the mortality distribution, in the J model the replacement ratio $\hat{\delta}$ is expressed as a weighted average of the vintage replacement ratios with weights given by the relative proportions of net investment of each age in the beginning of period capital stock.

From the above it follows that, with the exception of their difference regarding the useful life of durable goods, which is unrelated to the puzzle under consideration, the two models are identical because they use the same rules, conventions, definitions and assumptions. Hence, even though the model was applied differently, i.e. by Feldstein and Rothschild (1974) to highlight the relationship of $r(t)$ to $\alpha_\upsilon(t)$ and s_υ and by Jorgenson (1974) to address the relationship of $\hat{\delta}$ to various distributions of δ_τ, the difference in their analytical approaches should reinforce rather than lead to conflicts in the results. So let us see whether this is indeed the case.

3. Main results

The authors employed their models to tackle two issues. These were, first, to obtain necessary and sufficient conditions under which $r(t) = r$ and $\hat{\delta} = \delta$, and, second, to assess the applicability of these conditions in real world situations. The plan here is to present the results that they obtained with regard to the former issue.

3.1 Feldstein and Rothschild (1974, pp. 397-399)

Observe from the middle column of the last row in Table 3 that the replacement ratio would be constant if: a) either $r(t)$ is independent of the age structure of the capital stock, $\alpha_\upsilon(t)$, or b) the latter assumes only certain limited values. Consider first the conditions for $r(t)$ to be independent of the age structure of the capital stock. This would transpire if:

$$r(t) = \sum_{\upsilon=1}^{V} \alpha_\upsilon(t)(1 - s_\upsilon) = r \text{ for all } \alpha_\upsilon(t) \geq 0 \text{ such that } \sum_{\upsilon=1}^{V} \alpha_\upsilon(t) = 1 . \tag{3}$$

On close inspection it is easy to ascertain that (3) would be satisfied if and only if:

$$s_\upsilon = s, \text{ for } \upsilon = 1, 2, \ldots, V . \tag{4}$$

Now from Table 3 it is seen that $s_V = 0$. Hence, (4) can hold only for $V = 1$ or $V = \infty$. In the former case, capital would last only for one period and the problem would become economically uninteresting. So the authors exclude it from further consideration. As for the later case, i.e. the case in which capital lasts forever, (4) implies constant exponential output decay. This proves:

***Theorem 1**. The necessary and sufficient condition for the replacement ratio to be independent of the age structure of the capital stock, and thus give rise to $r(t) = r$, is that all capital must deteriorate at the same constant exponential rate.*

If output deterioration is not exponential, in order for the replacement ratio to be constant, the age structure of the capital stock must remain unchanged throughout the horizon of the renewal process. So what these authors investigated next was the conditions under which $\alpha_\upsilon(t)$ remains constant. In doing so they focused on the solution of the equation:

$$M(t+1) == B[q(t)] \cdot M(t)\,. \tag{5}$$

where $q(t)$ is the ratio of gross investment to the capital stock, called expansion coefficient, and B is a matrix given by:

$$B[q(t)] = \begin{bmatrix} q(t) & q(t) & q(t) & \cdots & q(t) \\ s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & s_{V-1} & 0 \end{bmatrix}. \tag{6}$$

To this effect, they applied two lines of analysis. In the first line they proved the following theorem:

***Theorem 2.** If a) $M(0) \geq 0$, $q(t) > 0$, $s_\upsilon > 0$* for υ=1,⋯,V-1, *and b) $q(t) = q$ for all t, there is a non-negative vector $E(q)$ such that $\|E(q) - a(t)\| \to 0$.*

This implies that, if the capital stock does not decay exponentially, $r(t)$ converges eventually to the constant r only in the very special case in which gross investment is a constant fraction of the capital stock and in which the capital stock eventually grows at a constant exponential rate.

In the second line of analysis their attention turned to the converse of the above theorem and the one below summarizes the results:

Theorem 3. *If a)* $M_1(t) \geq 0$ for all t, *b)* $0 < q \leq q(t) \leq \bar{q}$ for all t, *and c)* $Lim_{t\to\infty}\alpha(t) = \alpha$, *then the sequence* $q(t)$ *converges.*

What it asserts is that the age structure of the capital stock is or tends to a constant only if the sequence of expansion coefficients also converges to a constant. By implication, once again, but in more important way, they ascertained that if the deterioration of capital is not exponential, the replacement ratio tends to a constant only in the very special case in which gross investment becomes a constant fraction of the capital stock and therefore in which the capital stock eventually grows at a constant exponential rate. With the above in mind, let us turn now to summarize the results that were obtained in the second study.

3.2 Jorgenson (1974, pp. 191-204)

As in the above case, Jorgenson investigated the conditions under which the sequence of vintage replacement ratios δ_τ, for τ=1,2,$\cdots$, converges to $\hat{\delta} = \delta$ for exponential and non-exponential output decay functions.[5] To illustrate the former case, he assumed that the decline in the relative efficiency of capital follows the geometric distribution:

$$d_\tau = (1-\delta)^\tau \text{ for } \tau\text{=0, 1, 2,...}; \tag{7}$$

Inserting (7) into the mortality distribution yields:

$$m_\tau = d_{\tau-1} - d_\tau = \delta(1-\delta)^{\tau-1}. \tag{8}$$

Next, using (8) in conjunction with the definitions of R_t and K_t gives:

$$R_t = \sum_{\tau=1}^{\infty} \delta(1-\delta)^{\tau-1} A_{t-\tau} \ , \tag{9}$$

$$K_t = \sum_{\tau=0}^{\infty}(1-\delta)^{\tau} A_{t-\tau} \ . \tag{10}$$

Consequently, the change in the capital stock may be written as:

$$\begin{aligned} K_t - K_{t-1} = A_t - R_t &= A_t - \sum_{\tau=1}^{\infty}\delta(1-\delta)^{\tau-1}A_{t-\tau} \\ &= A_t - \delta K_{t-1}. \end{aligned} \tag{11}$$

This proves that the replacement ratio is equal to δ.

Next, he went on to investigate the more general case of non-geometric mortality distributions. He did so in four regimes involving: a) a single investment with fixed capital; b) multiple investments with fixed capital; c) a single investment with changing capital, and d) multiple investments with changing capital.[6] The results are summarized in Table 4 below. From them it turns out that in all cases the replacement ratio is or tends to

Table 4 Results for non-geometric mortality distributions

	Single investment	Multiple investments
Constant capital	*If sequence* $\{\delta_\tau\}$ *is non-periodic:*[1] $\delta_\tau = \frac{1}{\mu}$ *If sequence* $\{\delta_\tau\}$ *has period* θ*:* $\delta_{\tau\theta} = \frac{\theta}{\mu}$	*If sequence* $\{\delta_\nu\}$ *is non-periodic:* $\delta_\nu = \frac{1}{\mu}$ *If sequence* $\{\delta_\nu\}$ *has period* θ*:* $\delta_{\nu\theta} = \frac{\theta}{\mu}$
Changing capital (Increasing or decreasing)	In all cases: Gross, net and replacement investment grow at the same constant rate. The sequence of vintage replacement ratios approaches a constant.	

Notes: 1. The symbol μ denotes the expected value of the time to replacement

a constant irrespective of the nature of the mortality distribution. Thus, drawing on these results, he concluded:

> ***Theorem 4.*** *Irrespective of whether: (a) the capital stock is fixed or changing, and, (b) it is periodic or not, the sequence of vintage replacement ratios* $\{\delta_\tau\}$ *approaches a constant fraction* δ *of capital stock for (almost) any mortality distribution and for any initial age distribution of the capital stock. The result that the replacement is a constant fraction of the capital stock, which holds exactly for the geometric distribution, holds*

asymptotically for (almost) any distribution."(p. 195)

Are theorems 1-3 different from theorem 4? For, if they are, their difference(s) might be attributed to flaws in the analyses. The answer is that all four theorems have been obtained consistently from the same model and that they complement and reinforce each other in asserting that the replacement ratio is a constant fraction of the capital stock or tends to such a constant, if and only if the ratio of gross investment to capital stock is or approaches a constant. But if so, how can we explain that: a) Jorgenson (1974), on the one hand, and Feldstein and Rothschild (1974), on the other, arrived at diametrically opposite conclusions as to its applicability; b) the papers by Zarembka (1975) and Brown and Chang (1976), on the basis of which the controversy might have been elucidated, went largely unnoticed, and, c) the theorem has come to dominate economic theory and econometric applications? The objective below is to shed some light on these questions.

4. Likely explanations for the triumph and invincibility of the theorem

Above we found that, although the theorem was derived from the same model by applying a basic result of renewal theory, the protagonists in the debate, themselves leading authorities in this field, adopted different views about its usefulness. This implies that the explanation for its wide acceptance and staying power must be sought in reasons other than the status of the personalities involved or the credibility of renewal theory. With these possibilities out of the way, the following ones come into the forefront.

4.1 Realism versus instrumentalism in economics

When Schumpeter (1954) was writing his monumental *History of Economic Analysis*, he characterized the method by which economists approach the study of economic phenomena as follows:

> " Economic theory… cannot indeed, any more than can theoretical physics, do without simplifying schemata or models that are intended to portray certain aspects of reality and take some things for granted in order to establish others according to certain rules of procedure. So far as our argument is concerned, the things (propositions) that we take for granted may be called indiscriminately either hypotheses or axioms or postulates or assumptions or even principles, and the things (propositions) that we think we have established by admissible procedure are called theorems" (p. 15).

This passage describes precisely the way theorems 1-4 were obtained. But it does not give any hint as to how economists select better over good "models" and "theorems" and explains why economic theorists who adopt this methodological approach have split mainly into two groups. The first group, called instrumentalists,[7] consists of those who maintain that the appropriate selection criterion is the ability to *predict* the phenomena to which "theorems" pertain, without regard to the empirical validity either of the "models" themselves or the "hypotheses or axioms or postulates or assumptions or even principles" on which they rest. As for the second group, called realists,[8] this comprises all those who place the emphasis on the ability of "models" and "theorems" thereof to *explain* as well as *predict* the phenomena under consideration. In turn, what this requires is that both the "models" and their "premises" must be empirically valid.[9]

Now, suppose in the context of the above methodological remarks that Jorgenson (1963; 1965; 1974) wished to explain *net* investment. Since the latter cannot be observed directly, the only way to factor it out from *gross* investment is to estimate replacement investment. In doing so it sufficed for him to invoke the theorem in order to *predict* its magnitude as a proportion of the outstanding capital stock. On the contrary, if he were interested to *explain* replacement investment, he would have shown some interest in the robustness of the theorem with respect to the assumptions in Table 2. But this was not the case. For Jorgenson and his associates the theorem was useful because it was as good a mechanism to predict replacement investment as any other. Otherwise they were indifferent because, as the following passage from Leontief (1971) reveals, instrumentalism was well on its way to overtaking realism in economics:

> "In the presentation of a new model, attention nowadays is usually centered on a step-by-step derivation of its formal properties. But if the author-or at least the referee who recommended the manuscript for publication-is technically competent, such mathematical manipulations, however long and intricate, can even without further checking be accepted as correct. Nevertheless, they are usually spelled out at great length. By the time it comes to interpretation of the substantive *conclusions*, the assumptions are easily forgotten. But it is precisely the empirical validity of these *assumptions* on which the usefulness of the entire exercise depends"(p. 2).

In short, according to this explanation, the doubts that Feldstein and Rothschild (1974) and others expressed regarding the applicability of the theorem had little chance to succeed because they were addressed from a realist perspective, which was going out of fashion. The example below corroborates firmly this conjecture by establishing that the theorem fails with respect to at least one of the basic assumptions of the model.

4.1.1 Impossibility of the theorem under technological change

A look around us would suffice to convince anybody that capital goods are very heterogeneous. This means that we employ innumerable categories of such goods and that within each category there are old and new ones, which are differentiated by the technological progress that they embody. By implication, the assumption in Table 2 that durable goods are homogenous may be relaxed in at least two ways. That is, first, by recognizing the existence of more than one categories of durable goods, which are replaced in a like-for-like fashion, and, second, by allowing durable goods to be replaced by ones that incorporate the most recent advances in science and technology. Feldstein and Rothschild (1974, p. 401) did investigate the former case and showed that in a two-sector model without technological change the required necessary and sufficient conditions for the aggregate replacement ratio to be constant are extremely unlikely to be met in reality. Moreover, their results were further ascertained by the impossibility theorem derived by Zarembka (1975) and the farfetched conditions of the possibility theorem by Brown and Chang (1976). That is why the following example purports to show that in the presence of technological change the theorem does not hold even in the one-sector model.

Consider an economy with a representative firm which consists of two lines of production, one constructing an intermediate durable good called capital solely by means of labor and another producing a final good by combining each unit of capital with one unit of labor. In year υ, the firm uses capital capable of producing $K_X(\upsilon)$ units of output X. Usage does not wear capital because its effects are exactly offset by maintenance. But from the one period to the next $K_X(\upsilon)$ becomes more productive because newer vintages embody the most recent advances in science and technology. So to capture the impact of technological change, let the productivity of $K_X(\upsilon)$ increase at the constant exogenous rate μ_X. Then newer vintages of capital would present a competitive advantage to other firms that might wish to enter into business. For this reason, assume that to deter potential entrants the firm reduces the price of X at the rate of technological change. The question that arises is: Would the theorem of proportionality hold in this economy?

To answer it, consider an adaptation of the model presented in Bitros (2008, 2009c).[10] More specifically, assume that the above firm operates as if to maximize the value of its net worth over an infinite number of investment cycles, each of which lasts as many periods as the useful life of its capital T_X. If so, following the analysis in the above mentioned papers, it can be shown that one of the necessary conditions that must be satisfied is given by:

$$\sigma e^{-\mu_X T_X} - \mu_X e^{-\sigma T_X} = (1+\beta\sigma)(\sigma-\mu_X)\,. \tag{12}$$

where σ is the rate of interest and β stands for the minimum labor required for building one unit of $K_X(\upsilon)$. From expression (12), but also from its graphic solution in Figure 1,

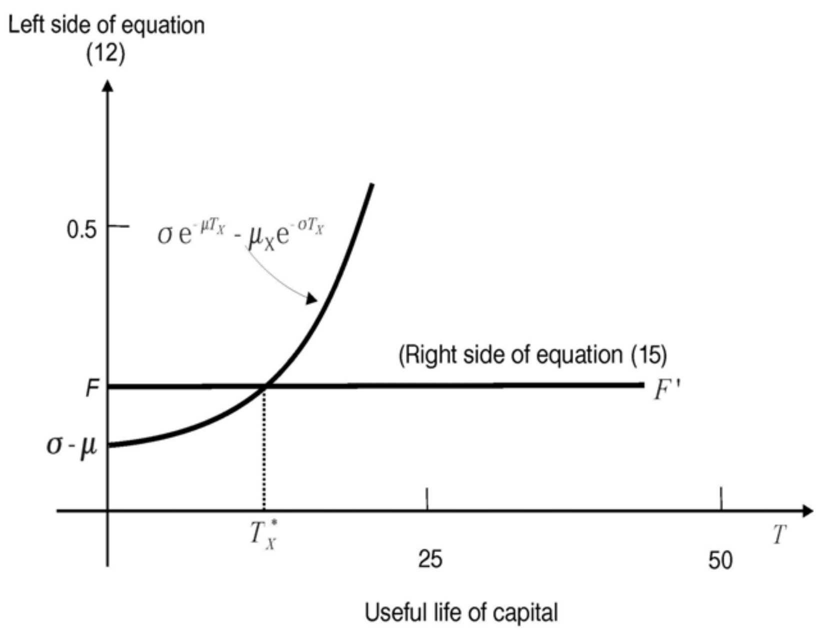

Figure 1

it follows that the useful life of capital T_X depends, among other economic influences, on the rate of technological change μ_X. This proves that the useful life of capital in this economy would not be invariant with respect to the rate of technological change and thus vitiates the theorem of proportionality.

4.3 Operational advantages of the theorem

Equation (7) gives the geometric distribution, which constitutes the discrete analog of the exponential one. Switching for convenience to the latter, the percentage of capital that survives to time τ is given by the so-called reliability function: $R(\tau) = \exp(-\delta\tau)$, for $\tau \geq 0$ and $\delta > 0$. Corresponding to the reliability function there is another function, $h(\tau)$, called hazard function or instantaneous failure rate function. The relationship between these two functions is $h(\tau) = -R'(\tau)/R(\tau)$, where the prime indicates the derivative of R. Thus in this case:

$$h(\tau) = \frac{\delta e^{-\delta\tau}}{e^{-\delta\tau}} = \delta. \tag{13}$$

Namely, the hazard function does not change over time. This is a unique property of the exponential distribution because it is the only one having a constant instantaneous failure rate. That is why we say that used means of production whose output efficiency deteriorates exponentially are as good as new or, otherwise, that the exponential distribution has no memory. On the contrary, if deterioration follows the reliability function: $R(\tau) = \exp(-\tau^2)$, then $-R'(\tau) = 2\tau \exp(-\tau^2)$ and $h(\tau) = 2\tau$. This implies that, the decline in output efficiency worsens linearly with time and used durable goods are not as good as new. This property indicates that the distribution underlying these reliability and hazard functions has memory.[11]

Viewed in the context of these remarks, the study of depreciation and replacement is far easier under exponential than non-exponential laws of deterioration. To corroborate it, recall from above that under exponential deterioration new units of capital are as efficient as used ones. This may be interpreted to imply that, while the quantity of capital units evaporates as by radioactive decay, the output efficiency of those that survive remains intact. As a result, since each surviving unit of capital has the same output efficiency, its age or durability or longevity or service life or useful life is immaterial and it may be ignored. In turn this yields a far-reaching simplification for the following four reasons: a) if all units of capital deteriorate at the same constant exponential rate, in the absence of embodied technological change, producer durables can be consistently aggregated into a measure of “capital-in-general” by invoking Theorem 1; b) the computation of capital stocks at any level of aggregation is greatly facilitated through the perpetual inventory method; c) as Hulten and Wykoff (1981) and Hulten, Robertson, and Wykoff (1989) have pointed out, using a single number to characterize the process of deterioration helps achieve “a major degree of simplification”, because it transforms a problem which is essentially non-stationary into a stationary one, and d) depreciation is dual to replacement and thus capital as a factor of production and as a measure of wealth coincide. All these advantages may explain why economic theorists and applied researchers have embraced the theorem with such unquestioned enthusiasm.

4.3 Availability of data

All publicly available information that has accumulated in the post-war period regarding stocks of fixed capital comes in the form of estimates obtained with the help of the so-called perpetual inventory method in conjunction with some assumption about the factor of proportionality, δ. Thus, if an empirically oriented economist wishes to acquire data on certain capital stock series for his research, the chances are that he will be able to get them or to construct them quickly and without much investment in time and resources. On the contrary, if he wishes to compute capital stock series on the basis of another methodology, say, like the one suggested by Prucha (1997), the task would require a significant diversion from the primary purpose of his investigation, and this only if he has the knowledge and the resources to accomplish it. What all this implies is that there is a built-in inertia in empirical research that favors the dominance of the theorem.

Moreover, this inertia is propagated further by the fact that changing over to a new approach would render obsolete much of the investment that has gone into the publication of capital stock series by national and international organizations. Certainly, if these data were produced in the private sector under competitive conditions, one would hope that at some point capital stock series based on a more fruitful approach would start to emerge and perhaps also supply might create its own demand. Yet under the present government driven system of producing and distributing such data, the rate of obsolescence of perpetual inventory based capital stock series is bound to be slow, if not nil. So this may be the hardest impediment to confront, if the incumbent theorem is to give way to one that would provide for an endogenously determined rate of depreciation.

4.4 Inertia of the status quo

How do economists come to believe what they believe, and to alter these beliefs over time? What part do empirical findings play in determining and affecting this web of beliefs? These are the two questions that Goldfarb (1997) posed and tried to elucidate by undertaking a detailed comparative assessment of the results in several fields of economics. At the end he concluded that:

> "The relative fragility of empirical findings suggested by the existence of so many 'emerging recalcitrant results' makes it more likely that theoretical preconceptions will be relatively impervious to empirical onslaughts."(p. 238)

But is the empirical evidence regarding the replacement ratio fragile? According to the assessment presented by Bitros (2009b), it is anything but fragile. More specifically, in the four decades from Jorgenson (1963) to Bu (2006) there appeared over 60 studies, which tested the theorem at different levels of aggregation using various methodological approaches, sets of data, and estimating techniques. From them not more that 5% might me classified as inconclusive, around 12% confirmed the theorem, whereas in the remaining 83% it was refuted with considerable degrees of confidence. From these figures it follows that the empirical evidence is overwhelmingly against the theorem and that, if this were the case in the hard sciences, the theorem would have been abandoned long ago. Hence, that this has not happened indicates that, aside from the processes already mentioned above, there may have been at work even stronger forces of inertia.

One of these forces may have been the view that the beliefs of economists are determined by theoretical considerations. Hirschman (1970, 67-68) introduced it into economics by drawing on the ideas about scientific revolutions advocated by Kuhn (1962). Its main argument is that a theory can be beat only by another theory, and not alone by "data". Or, expressed differently, a theory is not set aside due to conflicts in its predictions with reality, but because another theory is in better alignment with experience. Therefore, perhaps, research efforts aimed at falsification of the theorem by reference to "data" would have proved more successful in preventing its dominance in mainstream economics, if they had been oriented towards building a model leading to a more fruitful theorem.

Another force may have been the way in which graduates of economics departments, particularly in the United Stated, are taught and advance their academic careers. A cursory view in the curricula of leading universities would suffice to reveal that they pay leap service to education in the methodology of science. In my years of graduate education one might chose methodology as one of his fields and even write his Ph.D. dissertation in this area. However, since then related courses have twiddled to extinction and mathematicians and engineers have taken over the education of academic economists, neglecting the concerns that previous generations of economic theorists expressed about the proper approaches to confirmation or refutation of theoretical propositions in economic research. Hence, drawing also on the findings by Goldfarb (1995), it is not unlikely that the bias towards neoclassical replacement theory in the education of academic economists and in the publication of their research papers by leading economics journals may have played a significant role in the survival of the theorem over the onslaught of the empirical evidence referred to above.

Lastly, a significant source of inertia may have been the lack in economics of an apparatus by which to keep track of the empirical refutations and confirmations of a theorem and combine them into an index of acceptance or rejection. Very illuminating in this regard are the following views that Koopmans (1979) expressed in his 1978 presidential address to the American Economic Association:

> "The "if … then … " statements are similar to those in the formal sciences. They read like logical or mathematical reasoning in the case of economic theory, and like applications of statistical methods in the case of econometric estimation or testing. The heart of substantive economics is what can be learned about the validity of the "ifs" themselves, including the premises discussed above. "Thens" contradicted by observation call, as time goes on, for modification of the list of "ifs" used. Absence of such contradiction gradually conveys survivor status to the "ifs" in question. So I do think a certain record of noncontradiction gradually becomes one of tentative confirmation. But the process of confirmation is slow and diffuse….
> I have not found in the literature a persuasive account of how such confirmation of premises can be perceived and documented. How do we keep track of the contradictions and confirmations? How do we keep the score of surviving hypotheses? And what are we doing in those directions…. Meanwhile, unresolved issues, sometimes important from the policy point of view, and mostly quantitative ones, drag on and remain unresolved. Do they have to?"(11-12)

The answer to the last question is that certainly important issues do no have to remain unresolved and this explains my research in Bitros (2009a; 2009b) regarding the replacement ratio. However, before economists acquire the mindset of scientists in the hard sciences, it will take a variety of changes along the lines suggested by Teixeira (2007).

4.5 Lack of a better model

The tasks Feldstein and Rothschild (1974) pursued were first to obtain necessary and sufficient conditions for the theorem of proportionality to hold, and, second, to establish that these conditions are unlikely to be met in reality. As indicated in the preceding subsection, perhaps their research efforts would have proved more successful if they had presented a model leading to another more fruitful theorem. Yet this was not their plan and the field remained without an appropriate model that would challenge the established orthodoxy. Therefore, given that the theorem of proportionality has survived the massive empirical evidence against it, the time is quite ripe to redirect research efforts towards building a model capable to *explain* as well as *predict* replacement investment; In other words, to expand on the efforts of researchers in the tradition of the classical theory of replacement.

The starting point in this endeavor is to recognize that from a methodological standpoint successful research in empirical sciences quite often involves reviewing an established model and dethroning its non-reliable assumptions. In the present case, Table 2 shows that the model from which the theorem derives is based not on one but at least on three such assumptions. Consequently, a model in which they would be relaxed has good prospects to make a significant contribution in the field. Working in this direction, Bitros (2008; 2009c) constructed a model in which all three assumptions are replaced by precepts much closer to reality. For example, in this model two types of capital heterogeneity replace the assumptions that capital is homogeneous and that there is no embodied technological change. The one type of heterogeneity distinguishes durable goods into two categories according to their use, whereas the other differentiates durable goods within each category on the basis of the amount of technological change that they embody. Its analysis has shown that the theorem of proportionality fails. Moreover, it is argued that the theorem is alien to the thinking of researchers in industrial organization and neighboring fields to economics that treat the durability of capital goods as a choice variable; It ignores several thorny conceptual and methodological issues and, perhaps most important, it may have restrained seriously the progress towards developing models of capital based on more general approaches to production.

On these grounds then it is concluded that the prospects for continued dominance of the theorem in contemporary economics have started to retreat; But not yet in applications mainly because of the lack of pertinent data.

5. Summary of findings and conclusions

On account of their assumptions, conventions and definitions, the models that Feldstein and Rothschild (1974), on the one hand, and Jorgenson (1974), on the other, adopted to investigate the nature of the replacement investment-capital stock ratio turned out to be identical. Moreover both were cast in the context of renewal theory. But the authors used them to highlight the issue from different analytical perspectives. In particular, whereas the former authors focused on the relationship of the replacement ratio to the age structure of the capital stock and how the process of deterioration affects it, the latter author addressed the implications for the replacement ratio of various distributions describing the decline in the output efficiency of the capital stock. Thus, given that the theorems derived from the model were complementary and reinforced each other, one would have expected the authors to arrive at roughly similar conclusions. Instead not only they reached diametrically opposite conclusions, but also in the controversy that erupted there prevailed that view, which was weaker in terms of conceptual and empirical foundations. In this light the task set in the present paper was to elucidate the reasons that may have been responsible for this puzzling outcome.

From the middle of the 1970s it was already known that the theorem of proportionality failed in the presence of durables goods that are heterogeneous in kind and in the amount of technological progress that they embody. However, despite this limitation and the fact that it had been derived in the context of renewal theory from a one-sector model with homogeneous and undifferentiated durable goods, the theorem came to be adopted in economic theory and econometric applications as if it applied in general. This outcome implies that there were other forces at work that propagated it, even in the face of voluminous empirical evidence showing that it does not apply in reality. Digging deeper into the forces that may have been responsible, it emerged that the dominance and staying power of the theorem contributed: a) the discernible shift from realism to instrumentalism in the methodology of economics; b) its operational advantages, since on the one hand it reduced an essentially non-stationary problem into a stationary one, whereas on the other it helped construct series of capital stocks using the perpetual inventory method; c) the data that accumulated on capital stock series at various levels of aggregation that facilitated research in various fields without having to face the difficulty of generating appropriate series from scratch; d) the inertia of the status quo, which is fed and sustained by the lack in economics of a process by which to decide when a proposition is in conflict with experience and should be replaced or revised, and e) the lack of model leading to a more useful theorem than the one under consideration. Therefore, in the light of these findings, the time has come for research efforts to be directed towards constructing and testing models in which useful life of capital is determined endogenously in the presence of embodied technological change.

Appendix

Contemporary methodological guidelines for research in the empirical sciences

Notwithstanding important disagreements among philosophers of science, what is accepted today as appropriate methodological approach to science can be laid down briefly in the following four principles:

Principle I. A scientific theory (physics, biology, economics, sociology, medicine but NOT mathematics, logic, philosophy and other non-empirical disciplines) must be empirically testable. It must be verifiable said the logical positivists in the 1930's, falsifiable as Popper (1935) insisted then and later. The two are not equivalent: there is an asymmetry between verification and falsification, but that need not bother us here. The important thing is that scientific theories must be empirically testable. We can call this principle, the principle of empirical accountability. No empirical accountability, no science. Instead of science you have metaphysics.

Principle II Some metaphysics is instrumentally useful. It can serve heuristically. One may engage in a 'metaphysical' research programme from which certain empirical hypotheses can be deduced. We may call this principle, the principle of scientific speculation or hypothesis construction. One can use experience or imagination or metaphysical ideas as background; Certainly experience, which serves as background knowledge; But not induction.

Principle III There is no induction. What we call induction is unwarranted generalization from a finite number of observations. Whenever you believe you are using inductive thinking, you are really engaged in an activity described in Principle II above. There are no neutral observations. They are always theory-laden (or theory-impregnated). They contain theory. So you can't use a number of supposedly neutral observations to form a universal theory.

Principle IV What this boils down to is that usually theories (hypotheses) in empirical sciences are to be compared, say T_1 (the old one) and T_2 (the newly proposed one) and we judge their merits and demerits using various criteria. If we opt for T_2 and decide to discard T_1, it will be because the newer one has greater explanatory and/or predictive (or 'postdictive') power.

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Endnotes

[1] So indifferent was Haavelmo regarding the usefulness of received replacement theory that he did no make even a single reference to the contributions by these writers.

[2] For the sake of historical accuracy it should be noted that at about the same period other leading contributors to the neoclassical theory of capital adopted various ad hoc approaches to modeling depreciation. For example, Solow (1956) ignored depreciation altogether, whereas Samuelson (1962) introduced proportionality on the grounds that:

> "To keep the alpha good homogeneous independently of age, one has to assume a force of mortality independent of age (or an exponential life table). This means that physical depreciation is always directly proportional to the physical stock of alpha, K_a: Depreciation equals δ_a times K_a where the average length of life of alpha is the reciprocal of the δ_a factor."(p. 197)

[3] At that time prevalent among economists was the view that the only way to beat a theory is by another theory, not by "data" alone. An exposition of the foundations of this view is found in Hirschman (1970, 67-68). However, in the following decades mainstream economists shifted to the view, which is consistent with Friedman's (1953) famous methodology essay, that theories stand or fall on the basis of their ability to predict what the data reveal.

[4] Moreover, it may be of some interest to mention that Jorgenson's (1974) ignored also the sharp criticisms of his arguments by Feldstein (1972/1974) .

[5] Actually Jorgenson used the geometric distribution. He did so on the grounds that he employed discrete analysis. Had he applied continuous analysis, he would have assumed that the decline in the relative efficiency of capital followed the exponential distribution. But the results would have been just the same.

[6] A single investment is defined as one completed all at once. On the contrary, multiple is an investment completed piecemeal over a certain period.

[7] Friedman (1953) introduced this approach into economics following the epistemologist Duhem (1908), who recommended using theories as instruments and without concern if they are true or if their assumptions are realistic. According to the latter, what is important is whether the predictions derived from theories match appearances (phainomena), thus implying that models are useful not as causal explanations, but 'as if' ways of highlighting what appears before us.

[8] Drawing the debate that took place in the American Economics Review in the 1960s and the subsequent appraisal by Caldwell (1982), one would be justified to conclude that in economics leading authorities in the group of realists were Machlup (1955; 1964) and Samuelson (1963; 1965).

[9] For a brief but more detailed account of the principles that guide contemporary research in the empirical sciences, see the Appendix.

[10] The two-sector model analyzed in these paper is much more general in the sense that it provides for two sources of capital heterogeneity. That is, capital that belongs in different categories, like say laths versus electricity generators, and capital that differs from one vintage to the next, like laths and electricity generators built in 2007 versus those built in 2008.

[11] In particular, the probability distribution function that underlies the reliability and hazard functions in this case is Weibull with *shape* and *scale* parameters equal to 2 and 1, respectively.